Representability of $\triangle$-Matroids Over GF(2)

A. Bouchet and A. Duchamp
Département de Mathématiques et Informatique
Université du Maine
72017 Le Mans, Cedex, France

Submitted by Richard A. Brualdi

ABSTRACT

Let $A = (A_{vw} : v, w \in V)$ be a symmetric binary matrix. For $W \subseteq V$, let $A[W] = (A_{vw} : v, w \in W)$. The set $\mathscr{F} = \{W : W \subseteq V,\ A[W]$ has an inverse or $W = \emptyset\}$ satisfies the following *symmetric exchange axiom* (*SEA*): for $F', F'' \in \mathscr{F}$, for $x \in F' \triangle F''$, there exists $y \in F' \triangle F''$ such that $F' \triangle \{x, y\} \in \mathscr{F}$. A $\triangle$-matroid is a pair $(V, \mathscr{F})$ with a finite set V and $\emptyset \neq \mathscr{F} \subseteq \mathscr{P}(V)$ satisfying (*SEA*). We characterize those $\triangle$-matroids which can be obtained, as above, by means of a symmetric binary matrix.

1. INTRODUCTION

A set system is a pair $S = (V, \mathscr{F})$ with a finite ground set V and a subset $\mathscr{F} \subseteq \mathscr{P}(V)$ of feasible sets. We assume that the reader is familiar with the basic facts of matroid theory. It is convenient for our purpose to consider a matroid as a set system $M = (V, \mathscr{B})$, where $\mathscr{B}$ is the base set of M. Thus $\mathscr{B}$ is nonempty and satisfies the following *exchange axiom*:

(EA) For $B', B'' \in \mathscr{B}$ and $x \in B' \setminus B''$, there exists $y \in B'' \setminus B'$ such that $B' \triangle \{x, y\} \in \mathscr{B}$.

We denote by $\triangle$ the symmetric difference operator. A $\triangle$-*matroid* is a set system $S = (V, \mathscr{F})$ with a nonempty set of feasible sets satisfying the following *symmetric exchange axiom*:

(SEA) For $F', F'' \in \mathscr{F}$ and $x \in \mathscr{F}' \triangle F''$, there exists $y \in F' \triangle F''$ such that $F' \triangle \{x, y\} \in \mathscr{F}$.

655 Avenue of the Americas, New York, NY 10010 0024-3795/91/$3.50

Clearly (EA) implies (SEA), so that any matroid is a Δ-matroid. We notice that x and y are not necessarily distinct in (SEA) when they are in (EA).

Δ-matroids and similar structures have been recently introduced with various applications. A. Dress and T. Havel [6], motivated by an abstract study of metric properties, introduced set systems which they called metroids, and it is proved by A. Bouchet, A. Dress, and T. Havel [4] that the class of metroids is equal to the class the Δ-matroids where the empty set is feasible. Δ-matroids and symmetric matroids (defined in Section 4) have been introduced by A. Bouchet [2] with applications to the Euler tours of a 4-regular graph and a generalization of the greedy algorithm. Also motivated by the greedy algorithm, R. Chandrasekaran and S. Kabadi [5] introduced pseudomatroids which are equivalent to Δ-matroids, and Liqun Qi [9] introduced ditroids, a generalization of symmetric matroids. The polyhedral interpretation of the greedy algorithm has been studied by M. Nakamura [10], who refers to an old paper of D. Dunstan and D. Welsh [7] on the same subject. Finally we point out the partition of the Grassmannian into strata which can be described in terms of matroids. Extending this partition to other homogeneous surfaces, I. M. Gel'fand and V. V. Serganova [8] introduced a very general structure called a (W, Q)-matroid which encompasses matroids, greedoids, and symmetric matroids.

Let $A = (A_{vw} : v, w \in V)$ be a square matrix with coefficients in a field Q. We say that A is *antisymmetric* if $A_{vw} = -A_{wv}$ for every $v, w \in V$ and $A_{vw} = 0$ for every $v \in V$ (this last condition is specified for the fields of characteristic 2). We say that A is *quasisymmetric* if there exists a function $\varepsilon : V \to \{-1, +1\}$ such that $\varepsilon(v) A_{vw} = \varepsilon(w) A_{wv}$ holds for every $v, w \in V$. In particular, if ε is constant, A is a *symmetric matrix*. If A is either an antisymmetric or quasisymmetric matrix, it is called a matrix of *symmetric type*. For every $X \subseteq V$ we let $A[X] = (A_{vw} : v, w \in X)$. By convention we consider $A[\varnothing]$ as a nonsingular matrix. Let $S(A) = (V, \{X : X \subseteq V,\ A[X]$ is nonsingular$\})$. The following property is proved in [3].

PROPERTY 1.1. *If A is a matrix of symmetric type, then $S(A)$ is a Δ-matroid.*

A *strong representation* of the Δ-matroid S is a matrix A of symmetric type over a field Q such that $S = S(A)$. A necessary condition for S to have a strong representation is that $\varnothing$ is feasible, and then we say that S is a *normal Δ-matroid*. Normality can be overridden in the following way.

For a set system $S = (V, \mathcal{F})$ and $X \subseteq V$ let $\mathcal{F} \Delta X = \{F \Delta X : F \in \mathcal{F}\}$. It is easy to verify that $\mathcal{F} \Delta X$ satisfies (SEA) if $\mathcal{F}$ does. Therefore if we let $S \Delta X = (V, \mathcal{F} \Delta X)$, then S and $S \Delta X$ are both Δ-matroids or not. In many respects these Δ-matroids have similar properties, so that they are said to be *equivalent*. In particular the following property is proved in [3]:

PROPERTY 1.2. *If two normal Δ-matroids are equivalent and one of them has a strong representation A over a field Q, then the other one has also a strong representation A' over Q. Moreover A and A' are either both antisymmetric or both quasisymmetric.*

The Δ-matroid S is said to be *representable* over the field Q if there exists a normal Δ-matroid S' which has a strong representation over Q. It is proved in [3] that a matroid is representable over a field Q in the usual sense if and only if it is representable over Q in the sense of Δ-matroid theory.

If A is an antisymmetric matrix, then any feasible set of $S(A)$ has an even cardinality (recall that a nonsingular antisymmetric matrix has an even order). We say that a Δ-matroid is *even* if the symmetric difference of any two feasible sets has an even cardinality. The other Δ-matroids are said to be *odd*. For example, any matroid is an even Δ-matroid because its feasible sets (the bases) are equicardinal. We notice that evenness is compatible with equivalence of Δ-matroids. Thus the representation by antisymmetric matrices is more especially adapted to even Δ-matroids.

We notice that a binary matrix B is of symmetric type if and only if it is symmetric. Moreover, B is antisymmetric if and only if its diagonal is null. Finally, $S(B)$ is even if and only if B is antisymmetric. These remarks obviously do not hold for a field of characteristic different from 2.

The purpose of this paper is to study the representability of Δ-matroids over GF(2). The main result (Section 3) is a characterization by excluded minors, which generalizes Tutte's theorem for binary matroids. Section 4 introduces a new notion of weak representability, which is motivated by the extension to Δ-matroids of the second main characterization of binary matroids: each connected line is on three circuits. This property has still a meaning in Δ-matroid theory, but it is characteristic of those Δ-matroids that are weakly representable on GF(2). It turns out that an even Δ-matroid is weakly representable over GF(2) if and only if it is representable over GF(2) in the sense defined above. For odd Δ-matroids representability over GF(2) implies weak representability over GF(2), but the converse is false.

2. MINORS

Let $S = (V, \mathscr{F})$ be a Δ-matroid. For $x \in V$, we let

$$\mathscr{F} \setminus x = \{F : F \subseteq V - x,\ F \in \mathscr{F}\}, \qquad S \setminus x = (V - x, \mathscr{F} \setminus x),$$

$$\mathscr{F}/x = \{F : F \subseteq V - x,\ F + x \in \mathscr{F}\}, \qquad S/x = \{V - x, \mathscr{F}/x\}.$$

It is easy to verify that $\mathscr{F} \setminus x$ and $\mathscr{F}/x$ satisfy (SEA). Therefore $S \setminus x$ (S/x) is a Δ-matroid if $\mathscr{F} \setminus x$ $(\mathscr{F}/x)$ is nonempty. If S is a matroid, then $S \setminus x$ (S/x) is the matroid obtained by deleting (contracting) x. We call $S \setminus x$ and S/x the *elementary minors* of S at x. A *minor* is obtained by taking successive elementary minors. The following property is easy to verify.

PROPERTY 2.1. *For any Δ-matroid $S = (V, \mathscr{F})$, $x \in V$, and $F \subseteq V$, we have*

$$(S \Delta F)/x = (S/x) \Delta F \quad \text{if} \quad x \notin F,$$

$$(S \Delta F)/x = (S \setminus x) \Delta (F - x) \quad \text{if} \quad x \in F,$$

$$(S \Delta F) \setminus x = (S \setminus x) \Delta F \quad \text{if} \quad x \notin F,$$

$$(S \Delta F) \setminus x = (S/x) \Delta (F - x) \quad \text{if} \quad x \in F.$$

PROPERTY 2.2. *If a Δ-matroid S has a representation A over Q, then any elementary minor T of S has a representation B over Q. Moreover B is antisymmetric (quasisymmetric) if A is antisymmetric (quasisymmetric).*

Proof. Let V be the ground set of S, and let $x \in V$. Following Property 2.1, if S' is a Δ-matroid equivalent to S, then the elementary minors of S at x are equivalent to the elementary minors of S' at x. Therefore we may assume that S is normal, and by Property 1.2 we also may assume that $S = S(A)$. Clearly

$$S(A) \setminus x = S(A[V - x]),$$

and so the property holds with $B = A[V - x]$ when $T = S \setminus x$. If there is no $F \in \mathscr{F}$ such that $x \in F$, then there is nothing to prove, for $T = S/x$; otherwise we consider such an F. Following Property 2.1, we have

$$[S(A)/x] \Delta (F - x) = [S(A) \Delta F] \setminus x.$$

Since $F \in \mathscr{F}$, $S(A) \Delta F$ is normal, and so by Property 1.2 there exists a strong representation A' of $S(A) \Delta F$. Thus we have

$$[S(A)/x] \Delta (F - x) = S(A') \setminus x = S(A'[V - x]),$$

and so it easily follows that the property holds with $B = A'[V - x]$. ∎

3. MINIMAL NONBINARY Δ-MATROIDS

A Δ-matroid S which has no representation (either antisymmetric or quasisymmetric) over a given field Q is said to be *minimal* with this property if every elementary minor of S has a representation (of the same type) over Q. As is the case in matroid theory, the minimal Δ-matroids nonrepresentable over Q are interesting because they are, following Property 2.2, the forbidden minors of the Δ-matroids representable over Q. The minimal Δ-matroids nonrepresentable over Q are preserved by equivalence, and so it is sufficient to search for one of them in each class.

For any set system $S = (V, \mathscr{F})$, let $\mathscr{F}_{(2)} = \{F \in \mathscr{F} : 0 \leqslant |F| \leqslant 2\}$.

PROPERTY 3.1. *If $S = (V, \mathscr{F})$ is a normal set system, there exists precisely one binary Δ-matroid $S' = (V, \mathscr{F}')$ such that $\mathscr{F}_{(2)} = \mathscr{F}'_{(2)}$.*

Proof. Let $A = (A_{vw} : v, w \in V)$ be the binary matrix defined as follows. For every $v \in V$ such that $\{v\} \in \mathscr{F}$ we let $A_{vv} = 1$; otherwise $A_{vv} = 0$. For every pair of distinct elements $v, w \in V$, we let $A_{vw} = 1$ if and only if one of the two following cases occurs: (1) $\{v, w\} \in \mathscr{F}$ and either $\{v\} \notin \mathscr{F}$ or $\{w\} \notin \mathscr{F}$, (2) $\{v, w\} \notin \mathscr{F}$ and $\{v\} \in \mathscr{F}$ and $\{w\} \in \mathscr{F}$. We verify that the property holds with $S' = S(A)$, and that A is actually unique. ∎

PROPERTY 3.2. *Let $S = (V, \mathscr{F})$ be a minimal nonbinary Δ-matroid which is normal, and let $S' = (V, \mathscr{F}')$ be the binary Δ-matroid such that $\mathscr{F}_{(2)} = \mathscr{F}'_{(2)}$. The following properties hold:*

(i) $\mathscr{F} \,\Delta\, \mathscr{F}' = \{V\}$;
(ii) *if S is even then* $|V| = 4$;
(iii) *if S is odd then* $|V| = 3$.

Proof. Since S' is binary and S is nonbinary, there exists $F \in \mathscr{F} \,\Delta\, \mathscr{F}'$. If $F \neq V$, we consider the minors $T = (F, \mathscr{G}) = S \setminus (V - F)$ and $T' = (F, \mathscr{G}') = S' \setminus (V - F)$. These two Δ-matroids are binary and normal. The equality $\mathscr{F}_{(2)} = \mathscr{F}'_{(2)}$ implies $\mathscr{G}_{(2)} = \mathscr{G}'_{(2)}$, so that $T = T'$ by Property 3.1, a contradiction, since $F \in \mathscr{G} \,\Delta\, \mathscr{G}'$. Thus (i) is proved.

We notice that $|V| > 2$, because $|V| \leqslant 2$ would imply $\mathscr{F} = \mathscr{F}_{(2)}$ and $\mathscr{F}' = \mathscr{F}'_{(2)}$, so that $\mathscr{F} = \mathscr{F}'$, a contradiction. This implies in particular that we cannot have $\mathscr{F}_{(2)} = \mathscr{F}'_{(2)} = \{\varnothing\}$, since otherwise we should have $\mathscr{F}' = \{\varnothing\}$ and $\mathscr{F} = \{\varnothing, V\}$, and (SEA) could not be satisfied for $\mathscr{F}$.

We prove that $|V| = 4$ if S is even. Since $\mathscr{F}_{(2)} = \mathscr{F}'_{(2)} \neq \{\varnothing\}$, we can find some $\{x, y\} \in \mathscr{F} \cap \mathscr{F}'$. The Δ-matroids $T = (V - x, \mathscr{G}) = (S/x)\Delta\{y\}$ and

$T' = (V - x, \mathscr{G}') = (S'/x)\Delta\{y\}$ are normal. Moreover T and T' are binary like S/x and S'/x. We have $\mathscr{G} \Delta \mathscr{G}' = \{V - x - y\}$. Therefore $T \neq T'$, and since these Δ-matroids are binary, we must have $\mathscr{G}_{(2)} \neq \mathscr{G}'_{(2)}$ by Property 3.1. This implies $V - x - y \in \mathscr{G}_{(2)} \Delta \mathscr{G}'_{(2)}$, so that $|V| = 4$.

We prove that $|V| = 3$ if S is odd. First we claim that we may assume that $\mathscr{F}_{(2)}$ contains a singleton. Otherwise the binary matrix A such that $S' = S(A)$ is antisymmetric, because $\mathscr{F}_{(2)} = \mathscr{F}'_{(2)}$. Therefore S' is even. Since $\mathscr{F} \Delta \mathscr{F}' = \{V\}$ and S is odd, V is the single odd feasible set of S. Let us replace S by $S \Delta V$. This is still a normal Δ-matroid, but now $\varnothing$ is the single even feasible set of S, so that for $\mathscr{F}_{(2)} \neq \{\varnothing\}$ it is necessary to have a singleton in $\mathscr{F}_{(2)}$. Thus we may consider some $\{x\} \in \mathscr{F}_{(2)}$. We have also $\{x\} \in \mathscr{F}'_{(2)}$ because $\mathscr{F}_{(2)} = \mathscr{F}'_{(2)}$. We consider the Δ-matroids $T = S/x$ and $T' = S'/x$. As above, T and T' are normal and binary. If we let $T = (V - x, \mathscr{G})$ and $T' = (V - x, \mathscr{G}')$, we have $\mathscr{G} \Delta \mathscr{G}' = \{V - x\}$. As above, we conclude that $\mathscr{G}_{(2)} \Delta \mathscr{G}'_{(2)} = \{V - x\}$, so that $|V| = 3$. ∎

To actually search for the minimal nonbinary Δ-matroids, we first generate the possible candidates for S' in Property 3.2, which is done by means of a strong representation A. Then we determine S by $\mathscr{F} = \mathscr{F}'\Delta\{V\}$, and S is actually a solution if and only if $\mathscr{F}$ satisfies (SEA). Finally we find five classes of equivalent Δ-matroids which are nonbinary and minimal with this property. One normal Δ-matroid in each class is listed below. The Δ-matroids S_1, S_2, S_3 are odd and defined on $V = \{1, 2, 3\}$, while S_4 and S_5 are even and defined on $V = \{1, 2, 3, 4\}$:

$S_1 = \{\varnothing, 12, 23, 31, 123\}$.

$S_2 = \{\varnothing, 1, 2, 3, 12, 23, 31\}$. Any Δ-matroid equivalent to S_2 is made of all the subsets of V, except one.

$S_3 = \{\varnothing, 2, 3, 13, 12, 123\}$. Any Δ-matroid equivalent to S_3 is made of all the subsets of V, except two complementary subsets.

$S_4 = \{\varnothing, 12, 13, 14, 23, 24, 34\}$. Any Δ-matroid equivalent to S_4 is made of all the subsets of V with a given parity, except one.

$S_5 = \{\varnothing, 12, 23, 34, 41, 1234\}$. The Δ-matroids equivalent to S_5 are those which are equivalent to the uniform matroid of rank 2 over four elements, which yields again the classical characterization of binary matroids by Tutte [11].

4. WEAK REPRESENTABILITY

For a set system $S = (V, \mathscr{F})$ we let $\mathrm{Up}(S) = (V, \{F : F$ is a maximal element of $\mathscr{F}$ ordered by inclusion$\})$. The following property easily follows from [2] [we recall our convention of identifying a matroid M with a set system $(V, \mathscr{B})$, where $\mathscr{B}$ is the base set of M].

PROPERTY 4.1. *A set system $S = (V, \mathscr{F})$ is a Δ-matroid if and only if* $\mathrm{Up}(S \Delta X)$ *is a matroid for every* $X \subseteq V$.

We say that a Δ-matroid $S = (V, \mathscr{F})$ is *weakly representable* over the field Q if the matroid $\mathrm{Up}(S \Delta X)$ is representable over Q for every $X \subseteq V$.

PROPERTY 4.2. *If a Δ-matroid is representable over a field Q, then it is weakly representable over Q.*

We will show that the preceding property is a consequence of the representability theory developed in [3] for symmetric matroids. We recall these results.

A finite set V' given with a partition Π into classes of cardinality 2 is called a *symmetric set*. For any $x \in V'$ we denote by $x^{\sim}$ the element of V' such that $\{x, x^{\sim}\} \in \Pi$. For $T \subseteq V'$ we let $\tilde{T} = \{x^{\sim} : x \in T\}$, and we say that T is a *subtransversal* if $T \cap \tilde{T} = \emptyset$. If $T \cup \tilde{T} = V'$ also holds, then T is called a *transversal*.

A *transversal system* is a set system $S' = (V', \mathscr{F}')$ where V' is a symmetric set and $\mathscr{F}'$ is a set of transversals of V'. The subsets of the feasible sets of $\mathscr{F}'$ are called the *independent sets* of S'. The subtransversals of V' which are not independent and minimal with this property are the *circuits* of S'. The *trace* of S' over a transversal V is $S' \cap V = (V, \{F' \cap V : F' \in \mathscr{F}'\})$.

The transversal system S' is called a *symmetric matroid* if $S = S' \cap V$ is a Δ-matroid for some transversal V or V'. For every other transversal W of V' there exists a uniquely determined subset $X \subseteq V$ such that $W = (V \setminus X) \cap X^{\sim}$, and it is easy to verify that $S \Delta X$ is isomorphic to $S' \cap W$ through the bijection $\beta_X : W \to V$ defined by $\beta_X(x^{\sim}) = x$ for $x^{\sim} \in X^{\sim}$ and $\beta_X(x) = x$ for $x \in V \setminus X$. Thus $S' \cap W$ is a Δ-matroid isomorphic to $S \Delta X$, and a symmetric matroid appears as a set system which encompasses an equivalence class of Δ-matroids. The following property is proved in [2]:

PROPERTY 4.3. *The circuits of the matroid* $\mathrm{Up}(S' \cap W)$ *are the circuits of the symmetric matroid S' which are included in W.*

The representations of matroids by chain groups, introduced by Tutte [11], are well suited to the representations of circuits. We recall this process and its adaptation to symmetric matroids. For a field Q and a finite set V, any element of $Q^V = \{A = (A_v : v \in V) : A_v \in Q\}$ is called a *chain* (on Q over V). The *support* of a chain A is $\|A\| = \{v \in V : A_v \neq 0\}$. Any subspace N of Q^V

is called a *chain group* (on Q over V). A nonnull chain of N having a minimal support is called an *elementary chain* of N. The supports of the elementary chains of N are the circuits of a matroid M, and N is called a *chain group representation of* M. Consider now a symmetric set V', a function $\varepsilon' : V' \to \{-1, +1\}$, and the bilinear form b_ε over $Q^{V'}$ defined for any two chains A and B by the formula

$$b_\varepsilon(A, B) = \sum \left(\varepsilon(v) A_v B_{\tilde{v}} : v \in V'\right).$$

We say that a chain group $N \subseteq Q^{V'}$ is *isotropic* if $A, B \in N \Rightarrow b_\varepsilon(A, B) = 0$. The supports of the elementary chains of N which are subtransversals constitute the set of the circuits of a symmetric matroid S' [3], and we call N a chain group representation of S'.

Given a Δ-matroid $S = (V, \mathscr{F})$ and a symmetric set V' which admits V as a transversal, we notice that the transversal system $S' = (V', (F \cup (V \setminus F)^{\sim} : F \in \mathscr{F}\})$ is the unique symmetric matroid defined on the carrier V' such that $S' \cap V = S$. It is equivalent for S to be representable over the field Q and S' to have a chain group representation over Q [3].

Proof of Property 4.2. We use the above notation. Suppose that the Δ-matroid S is actually representable over Q. Then the symmetric matroid S' has a chain group representation N over Q. Let $X \subseteq V$, $W = (V \setminus X) \cup X^{\sim}$, and N_W be the canonical projection over Q^W of the chains of N whose supports are included in W. Then, following Property 4.3, N_W is a chain group representation of the matroid $\mathrm{Up}(S' \cap W)$. The matroid $\mathrm{Up}(S \Delta X)$ is the isomorphic image of $\mathrm{Up}(S' \cap W)$ through β_X, and so it is representable over Q. ■

REMARK 4.4. It is possible to define a circuit of the Δ-matroid S directly, without constructing a symmetric matroid S' such that $S' \cap V = S$. Consider an ordered pair (P, Q) of disjoint subsets P and Q included in V, and say that (P, Q) is *separable* if there exists a feasible set F satisfying $P \subseteq F$ and $Q \cap F = \varnothing$. It is easy to verify that (P, Q) is separable if and only if $P \cup Q^{\sim}$ is an independent set of S'. Say that (P, Q) is a circuit of S if (P, Q) is nonseparable and minimal with this property [which means that every (P', Q') satisfying $P' \subseteq P$ and $Q' \subseteq Q$ with at least one strict inclusion is independent]. Then (P, Q) is a circuit of S if and only if $P \cup Q^{\sim}$ is a circuit of S'.

The symmetric matroid S' will be said to be weakly representable over Q if the matroid $\mathrm{Up}(S' \cap W)$ is representable over Q for every transversal W. It is equivalent to say that the Δ-matroid $S' \cap V$ is weakly representable over Q.

A *connected line* L of a matroid M is a minimal set which is the union of two nondisjoint circuits of M. Tutte proved that M is a binary matroid if and only if every connected line includes precisely three circuits. We define a connected line L of the symmetric matroid S' as a minimal set which is subtransversal and equal to the union of two nondisjoint circuits. Thus L is contained in some transversal W of V', and this amounts to saying that L is a connected line of the matroid $\mathrm{Up}(S' \cap W)$. Following this remark, it is easy to verify the following property.

PROPERTY 4.5. *A symmetric matroid is weakly binary if and only if every connected line includes precisely three circuits.*

Using Remark 4.4, it is easy to adapt this characterization to the weak representability of $\triangle$-matroids.

5. WEAKLY BINARY $\triangle$-MATROIDS

Let $S = (V, \mathscr{F})$ be a $\triangle$-matroid. For $v \in V$ we easily verify the equalities $\mathrm{Up}(S \setminus v) = \mathrm{Up}(S) \setminus v$ if v is not an isthmus of $\mathrm{Up}(S)$, $\mathrm{Up}(S \setminus v) = \mathrm{Up}(s)/v$ if v is an isthmus of $\mathrm{Up}(S)$, and $\mathrm{Up}(s/v) = \mathrm{Up}(S)/v$. Therefore, if S is weakly representable over a field Q, then every minor of S is also weakly representable over Q. The definition of weak representability also implies that every $\triangle$-matroid equivalent to S is also weakly representable over Q. Suppose that we know a set $\mathscr{E}_Q$ of matroids which are not representable over Q (a set of excluded minors) and such that every matroid nonrepresentable over Q has a minor isomorphic to a matroid belonging to $\mathscr{E}_Q$. The preceding remarks imply that for every $\triangle$-matroid S which is not weakly representable over Q, we can find a minor s of S and a $\triangle$-matroid t equivalent to s such that $\mathrm{Up}(t)$ is isomorphic to an excluded minor. The bases of the matroid $\mathrm{Up}(t)$ are the maximal feasible sets of the $\triangle$-matroid t. Thus t can be reconstructed from $\mathrm{Up}(t)$ by taking each base of $\mathrm{Up}(t)$ as a feasible set and eventually adding supplementary feasible sets taken among the independent sets of $\mathrm{Up}(t)$. Obviously this must be done in such a way that (SEA) holds. Anyway the number of possibilities for reconstructing t from $\mathrm{Up}(t)$ is finite. Thus we have the following property.

PROPERTY 5.1. *If there is a finite set of excluded minors for the representability of a matroid over a field Q, then there is a finite set of excluded minors for the weak representability of a $\triangle$-matroid over Q.*

This result can be applied to $Q = \mathrm{GF}(2)$ and $Q = \mathrm{GF}(3)$ by using the results of Tutte [11] and Reid [1] respectively. We have no method other than brute force to actually enumerate the possible candidates for t when $\mathrm{Up}(t)$ is known, which has little interest in general. However this enumeration can easily be done for $Q = \mathrm{GF}(2)$, because U_2^4, the uniform matroid of rank 2 over four elements, is the single excluded minor.

First let us enumerate a minimal set $\mathscr{A}$ of excluded minors for the weak representability of an even Δ-matroid over GF(2). To reduce the enumeration we assume that $\mathscr{A}$ does not contain any pair of equivalent Δ-matroids. Notice that a minor of an even Δ-matroid is even, so that $\mathscr{A}$ contains only even Δ-matroids. To obtain an even Δ-matroid t whose set of feasible sets is the base set of $\mathrm{Up}(t) = U_2^4$ eventually augmented by some independent sets, there are only two possibilities: to add no independent set or to add the empty independent set. The first possibility is U_2^4, denoted S_5 in the list enumerated in Section 3, and the second possibility yields a Δ-matroid equivalent to S_4. These two Δ-matroids are, up to equivalence, the excluded minors for the representability of an even Δ-matroid over GF(2). Since representability implies weak representability by (4.2), we obtain the following property.

Property 5.2. *An even Δ-matroid is representable over* GF(2) *if and only if it is weakly representable over* GF(2).

This property is a particular case which can neither be extended to the representability of an odd Δ-matroid over GF(2) nor be extended to the representability of an even Δ-matroid over another field.

For the representability over GF(2) we already know of the existence of an odd Δ-matroid S of order 3 which is nonrepresentable (take S_1, S_2, or S_3). No minor t of S can be such that $\mathrm{Up}(t) = U_2^4$, because U_2^4 has order 4, so that S is weakly binary. An enumeration of the excluded minors, up to equivalence, for the representability over GF(2) of an odd Δ-matroid yields, apart from S_4 and S_5, six new Δ-matroids, which are listed below with their feasible sets:

$$\{\varnothing, 1, 12, 13, 14, 23, 24, 34\}$$
$$\{\varnothing, 1, 2, 12, 13, 14, 23, 24, 34\}$$
$$\{1, 2, 3, 12, 13, 14, 23, 24, 34\}$$
$$\{\varnothing, 1, 2, 3, 12, 13, 14, 23, 24, 34\}$$
$$\{1, 2, 3, 4, 12, 13, 14, 23, 24, 34\}$$
$$\{\varnothing, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34\}.$$

For the representability over another field we consider GF(3) and the binary Δ-matroid $S = S(A)$, where A is the binary antisymmetric matrix indexed on $V = \{\theta, a, b, c, d, e\}$ depicted in Figure 1. Notice that A is the adjacency matrix of the 5-wheel, W_5, depicted in Figure 2.

$$\begin{array}{c} \\ \theta \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{cccccc} \theta & a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

FIG. 1.

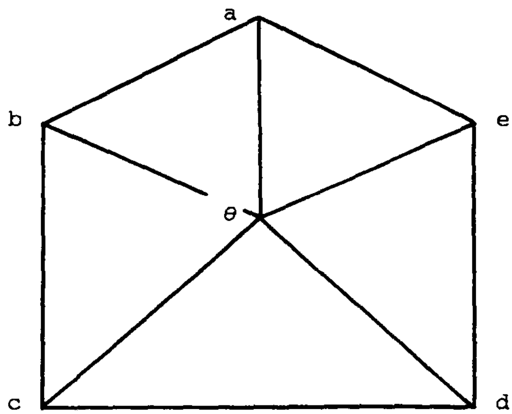

FIG. 2.

PROPERTY 5.3. *The Δ-matroid S is nonrepresentable over* GF(3), *but it is weakly representable over* GF(3).

Proof. Suppose that S is representable over GF(3). Then we can change each binary value 1 of A into a value $+1$ or -1 in GF(3), to obtain a new antisymmetric matrix A' with entries in GF(3) such that $S = S(A')$. This "signing" induces in a natural way an orientation of W_5 (an edge xy is oriented from x to y if and only if the entry $A'_{xy} = +1$). Consider an element $x \in V$, and change the signs in the row and in the column of A' indexed by x. This does not change any determinant of a submatrix $A[X]$, $X \subseteq V$, so that $S(A')$ remains unchanged. The effect is to reverse the orientations of the edges incident to x. Then it is easy to verify that we can successively perform the preceding operation on vertices among a, b, c, d, e so that the cycle (a, b, c, d, e) becomes consistently oriented, which is now assumed.

Consider the subset of vertices $X = \{\theta, a, b, c\}$. The submatrix $A[X]$ has a null determinant in GF(2). For $A'[X]$ to have a null determinant in GF(3) it is necessary that one of the two edges θa and θc has its head equal to θ while the other one has its tail equal to θ. This property must also hold for the sets of vertices $\{\theta, b, c, d\}$, $\{\theta, c, d, e\}$, $\{\theta, d, e, a\}$, $\{\theta, e, a, b\}$. We verify that these five conditions are incompatible. Therefore S is nonrepresentable over GF(3).

For every $X \subseteq V$ the matroid $\mathrm{Up}(S \triangle X)$ is binary because S is binary. The Fano matroid and its dual have order 7, so that neither of them can be a minor of $\mathrm{Up}(S \triangle X)$. Thus Tutte's characterization of regular matroids by excluded minors [11] implies that $\mathrm{Up}(S \triangle X)$ is a regular matroid. In particular $\mathrm{Up}(S \triangle X)$ is representable over GF(3). Therefore S is weakly representable over GF(3). ■

REFERENCES

1 R. E. Bixby, On Reid's characterization of the ternary matroids, *J. Combin. Theory Ser. B* 26:174–204 (1979).
2 A. Bouchet, Greedy algorithm and symmetric matroids, *Math. Programming* 38:147–159 (1987).
3 ——, Representability of $\triangle$-matroids, in *Colloq. Math. Soc. János Bolyai* 52, *Combinatorics*, Eger, Hungary, 1987, pp. 167–182.
4 A. Bouchet, A. Dress, and T. Havel, $\triangle$-matroids and metroids, *Adv. Math.* In press.
5 R. Chandrasekaran and S. N. Kabadi, Pseudomatroids, *Discrete Math.* 71:205–217 (1988).
6 A. Dress and T. Havel, Some combinatorial properties of discriminants in metric vector spaces, *Adv. Math.* 62:285–312 (1986).
7 F. D. J. Dunstan and D. J. A. Welsh, A greedy algorithm for solving a certain class of linear programming, *Math. Programming* 5:338–353 (1973).
8 I. M. Gel'fand, Combinatorial geometries and torus strata on homogeneous compact manifolds, *Russian Math. Surveys* 42(2):133–168 (1987).
9 Liqun Qi, Directed submodularity, ditroids and directed submodular flows, *Math. Programming* 42:579–599 (1988).
10 M. Nakamura, A characterization of greedy sets: Universal polymatroids (I), *Sci. Papers College Arts Sci. Univ. Tokyo* 38:155–167 (1988).
11 W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Standards B* 69:1–47 (1965).

Received 16 October 1989; final manuscript accepted 20 June 1990